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# On a Class of Exactly Soluble Statistical Mechanical Models with Nonpolynomial Interactions

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The approximating Hamiltonian method of N. N. Bogolubov, Jr. is generalized to models with nonpolynomial intensive-observable interactions. The original Hamiltonian is proved to be thermodynamically equivalent to one linear in the intensive-observable trial Hamiltonian. We show that the exact expression for the free energy density in the thermodynamic limit can be obtained from a min-max principle for the system with trial Hamiltonian.

**KEY WORDS:** Approximating Hamiltonian method; nonpolynomial interactions; intensive observables; thermodynamic equivalence.

## 1. INTRODUCTION

In our previous paper[1] we proposed a further extension of the approximating Hamiltonian method of Bogolubov, Jr.[2,3] which permits the asymptotically exact (i.e., exact in the thermodynamic limit) investigation of a general class of model systems with a nonpolynomial interaction term. The interaction is a function of the space average of some quasilocal operator (observable; see Haag[4] and the Appendix). Thus it is a function of an intensive observable of the system. In the case under consideration, the $N$-body Hamiltonian, defined in a region $\Lambda \subset \mathbb{R}^\nu$ ($\nu = \dim \mathbb{R}^\nu$) with a finite volume $|\Lambda|$, acts on the Hilbert space of states $\mathfrak{H}_\Lambda$ and has the form[4]

$$H_\Lambda = T_\Lambda - h|\Lambda|A_\Lambda - |\Lambda|\varphi(A_\Lambda) \tag{1.1}$$

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[4] For other types of nonpolynomial models (generalized Dicke-type models) see Ref. 5.

Here $h \in \mathbb{R}^1$, $T_\Lambda$, and $A_\Lambda$ are *self-adjoint* operators satisfying the following conditions:

(i) $A_\Lambda$ is an *intensive observable* generated by the space averaging of some self-adjoint, quasilocal operator uniformly bounded in $\mathfrak{H}_\Lambda$, i.e., there exists $M > 0$ such that for all $\Lambda \subset \mathbb{R}^\nu$, with $|\Lambda| < \infty$

$$\|A_\Lambda\|_{\mathfrak{H}_\Lambda} \leqslant M$$

(ii) The operator $T_\Lambda$, which generally defines an *extensive observable* of the system, is such that there exists $K' > 0$ satisfying

$$\|[T_\Lambda, A_\Lambda]_-\|_{\mathfrak{H}_\Lambda} \leqslant K'$$

for all $\Lambda \subset \mathbb{R}^\nu$ with $|\Lambda| < \infty$.

(iii) The operator-valued function $\varphi(A_\Lambda)$ can be defined by the spectral representation

$$\varphi(A_\Lambda) = \int_{-M}^{M+0} dE_\lambda(A_\Lambda)\, \varphi(\lambda)$$

where $\varphi(\lambda)$ is a twice differentiable function on $\Pi = [-M, M]$ [$\varphi(\lambda) \in C^2(\Pi)$] such that there exists $K > 0$ and the following inequality holds:

$$|\varphi''(\lambda)| \leqslant K$$

Without loss of generality we further assume $\varphi(0) = \varphi'(0) = 0$ [see (1.1)].

(iv) The operator $T_\Lambda$ generates the Gibbs semigroup $\{\exp(-\beta T_\Lambda)\}_{\beta>0}$, i.e., $\exp(-\beta T_\Lambda) \in$ Trace-class for all $\beta > 0$.

(v) By virtue of conditions (i) and (iv), the operator

$$\Gamma_\Lambda(x) = T_\Lambda - x|\Lambda|A_\Lambda, \qquad x \in \mathbb{R}^1 \tag{1.2}$$

also generates the Gibbs semigroup; we require the existence of the thermodynamic limit $t\text{-lim}(\cdot) \equiv \lim_{|\Lambda|/N = v, N\to\infty, |\Lambda|\to\infty}(\cdot)$ (where $|\Lambda| \to \infty$ in the sense of Fisher[6]) for the free energy density

$$F_\Lambda(x) = -(\beta|\Lambda|)^{-1} \ln \operatorname{Tr} \exp[-\beta\Gamma_\Lambda(x)] \tag{1.3}$$

namely, for all $x \in \mathbb{R}^1$, $\beta > 0$, $v > 0$ there exists a function such that

$$t\text{-lim}\, F_\Lambda(x) = F(x), \qquad F_\Lambda(x) \in C^\infty(\mathbb{R}^1) \tag{1.4}$$

(vi) Define the *approximating Hamiltonian*

$$H_{0,\Lambda}(a) = \Gamma_\Lambda(h + \varphi'(a)) + |\Lambda|(a\varphi'(a) - \varphi(a)) \tag{1.5}$$

which depends on the real parameter $a \in \Pi$; for the system with Hamiltonian (1.5) and all $\beta > 0$ and $v > 0$ the following *clustering property* must hold:

$$t\text{-lim}\{\langle A_\Lambda{}^2\rangle_{H_{0,\Lambda}(\bar{a}_\Lambda)} - \langle A_\Lambda\rangle^2_{H_{0,\Lambda}(\bar{a}_\Lambda)}\} = 0 \tag{1.6}$$

where $\bar{a}_\Lambda$ is determined from the equation

$$\min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)] = f_\Lambda[H_{0,\Lambda}(\bar{a}_\Lambda)], \qquad S_\Lambda = \{a \in \mathbb{R}^1\colon \ a = \langle A_\Lambda \rangle_{H_{0,\Lambda}(a)}\} \tag{1.7}$$

Here use has been made of the notation

$$\langle\cdot\rangle_H = \mathrm{Tr}\{(\cdot)\exp(-\beta H)\}/\mathrm{Tr}\exp(-\beta H)$$
$$f_\Lambda[\cdot] = -(\beta|\Lambda|)^{-1}\ln \mathrm{Tr}\exp\{-\beta(\cdot)\}$$

for the thermal average and the free energy density, respectively.

**Remark 1.1.** The clustering condition (1.6) corresponds to certain restrictions on the magnitude of the fluctuations of the intensive observable $A_\Lambda$ in the system described by the approximating Hamiltonian (for further details see the Appendix).

**Proposition 1.1.**[1] Let the Hamiltonian of the system be given by Eq. (1.1) and let conditions (i)–(vi) be satisfied; then

$$t\text{-}\lim \left| f_\Lambda[H_\Lambda] - \min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)]\right| = 0 \tag{1.8}$$

where $H_{0,\Lambda}(a)$ has been defined by Eq. (1.5).

**Remark 1.2.** As we have shown in Ref. 1,

$$\min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)] = \min_{a\in\mathbb{R}^1}\max_{b\in\mathbb{R}^1} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)] \tag{1.9}$$

where

$$\mathscr{H}_{0,\Lambda}(a, b) = \Gamma_{0,\Lambda}(h + \Phi_1'(a) + \Phi_2'(b)) + |\Lambda|\{a\Phi_1'(a) - \Phi_1(a) + b\Phi_2'(b) - \Phi_2(b)\} \tag{1.10}$$

Here

$$\Phi_1(a) = \tilde{\varphi}(a) + \tfrac{1}{2}La^2, \qquad \Phi_2(b) = -\tfrac{1}{2}Lb^2 \qquad (L > 3K) \tag{1.11}$$

and the function $\tilde{\varphi}(a) \in C^2(\mathbb{R}^1)$ is a twice differentiable extension of $\varphi(a) \in C^2(\Pi)$ to $\mathbb{R}^1$, which satisfies condition (iii).

In the general case [this means that approximating Hamiltonian (1.5) is not to be one-particle operator] the direct calculation of the thermodynamic limit $t$-$\lim\{\min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)]\}$ is hardly practicable because of the absence of an explicit expression for $f_\Lambda[H_{0,\Lambda}(a)]$ at large but finite values $N$ and $|\Lambda|$ as well as because of the lack of explicit information about the structure of the set $S_\Lambda$ for $|\Lambda| \to \infty$. In the present paper it will be shown how to avoid these difficulties provided the limit function [see (v)]

$$t\text{-}\lim f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)] = F(h + \Phi_1'(a) - Lb) - \tfrac{1}{2}Lb^2 + a\Phi_1'(a) - \Phi_1(a) \tag{1.12}$$

is known.

**Remark 1.3.** Simultaneously with our work,[1] the same problem has been studied by den Ouden *et al.*[7] They have considered the same Hamiltonian as (1.1) but containing an *analytic function* of a finite number of intensive (normalized[7]) self-adjoint operators $A_\Lambda^{(i)}$, $i = 1, 2,..., n$, under restrictions stronger than (i)–(vi) on the operators $T_\Lambda$ and $A_\Lambda^{(i)}$ and the function $\varphi(\cdot)$ (see below, Remark 1.4). In a recent preprint[8] the same authors have given a convex-envelope formulation of the problem in the fixed-magnetization ensemble.[5]

This paper presents a further development of the approach proposed in Ref. 1 for systems with nonpolynomial interactions. In particular we shall give here a complete proof (see Sections 2 and 3) of the fact

$$t\text{-lim}\left\{\min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)]\right\} = \min_{a\in S}\{t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]\} \qquad (1.13)$$

which is important for practical applications of Proposition 1.1. In Eq. (1.13) the set $S$ is defined by inequalities (1.14) (see below), which, as was first shown in Ref. 7, replace the usual *self-consistency* equations (*molecular-field* equations). Below, a new derivation of Eq. (1.14) is given which is based entirely on the analysis of the auxiliary two-parameter variational problem for the limit function (1.12). The important particular cases of *attractive* $[\varphi''(a) > 0]$ and *repulsive* $[\varphi''(a) < 0]$ interactions are also paid special attention (see Section 3). The main result of the present paper can be formulated as follows:

**Theorem 1.1.** Let the Hamiltonian of the system be given by Eq. (1.1) and let the operators $T_\Lambda$ and $A_\Lambda$ and the function $\varphi(\cdot)$ satisfy conditions (i)–(vi); then:

(a) $t\text{-lim}\, f_\Lambda[H_\Lambda]$ exists for all $h \in \mathbb{R}^1$, $\beta > 0$, and $v > 0$.
(b) $t\text{-lim}\, f_\Lambda[H_\Lambda] = \min_{a\in S}\{t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]\}$, where

$$S = \{a \in \mathbb{R}^1\colon -F'(h + \varphi'(a) - 0) \leqslant a \leqslant -F'(h + \varphi'(a) + 0)\} \qquad (1.14)$$

**Remark 1.4.** The above theorem is a generalization of the result obtained in Refs. 7 and 8 under the condition that function is analytic on $\Pi$ and the operators $T_\Lambda$ and $A_\Lambda$ satisfy certain "short-range" conditions. We extend this result to the case of the broader class of functions $\varphi(a) \in C^2(\Pi)$ [(iii)] and reduce the restrictions on the range of interactions included in $T_\Lambda$

[5] The convex-envelope construction has been proposed by Lebowitz and Penrose[9] for a mathematically rigorous derivation of the van der Waals equation for classical gases with a long-range, Kac-type potential (see also Ref. 10). Generalization to quantum systems has been obtained by Lieb[11] (for further generalizations see Ref. 12 and also the review article, Ref. 13).

and $A_\Lambda$ to the more general conditions (iv)–(vi). In particular, we do not need the boundedness of the intensive (normalized[7]) operator $|\Lambda|^{-1}T_\Lambda$. Thus $T_\Lambda$ may correspond, for example, to the kinetic energy operator of particles enclosed in a region $\Lambda \subset \mathbb{R}^\nu$.

The proof of Theorem 1.1 follows a line of reasoning different from Ref. 7 and is based essentially on Proposition 1.1 and the main Lemma 2.1 (see Section 2). The idea of our proof consists in the consecutive establishing of the following four relations:

(1) $$t\text{-lim}\left| f_\Lambda[H_\Lambda] - \min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)]\right| = 0$$

(2) $$\min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)] = \min_{a\in\mathbb{R}^1}\max_{b\in\mathbb{R}^1} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]$$

(3) $$t\text{-lim}\left\{\min_{a\in\mathbb{R}^1}\max_{b\in\mathbb{R}^1} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\right. = \min_{a\in\mathbb{R}^1}\max_{b\in\mathbb{R}^1}\{t\text{-lim} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\}$$

(4) $$\min_{a\in\mathbb{R}^1}\max_{b\in\mathbb{R}^1}\{t\text{-lim} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\} = \min_{a\in S}\{t\text{-lim} f_\Lambda[H_{0,\Lambda}(a)]\}$$

Equations (1) (Proposition 1.1) and (2) have been obtained in Ref. 1 and the proofs of Eqs. (3) and (4) are given respectively in Sections 2 and Section 3 of the present paper. The combination of equalities (2)–(4) gives (1.13) and of (1)–(4) gives the statement (b) of Theorem 1.1.

## 2. THE MAIN LEMMA

We start with the proof of Eq. (3) (see Section 1), which is the content of the following main lemma:

**Lemma 2.1.** Let $\{f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\}$ be a sequence of functions generated by the two-parameter family of Hamiltonians (1.10) with operators $T_\Lambda$ and $A_\Lambda$ satisfying conditions (i)–(vi) (Section 1). Then:

(a) $\min_{a\in\mathbb{R}^1}\max_{b\in\mathbb{R}^1}\{t\text{-lim} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\} = f(\beta, v, h)$ exists for all $h \in \mathbb{R}^1$, $\beta > 0$, and $v > 0$.

(b) $f(\beta, v, h) = t\text{-lim}\{\min_{a\in\mathbb{R}^1}\max_{b\in\mathbb{R}^1} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]$.

*Proof.* (*a*) Let us denote $z = h + \Phi_1'(a) - Lb$. Then [see (1.3) and (1.10)] one has

$$f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)] = F_\Lambda(z) + a\Phi_1'(a) - \Phi_1(a) - \tfrac{1}{2}Lb^2 \qquad (2.1)$$

Conditions (i) and (v) (Section 1) imply the *uniform equicontinuity* of the family $\{F_\Lambda(x)\}$, since

$$|F_\Lambda(x') - F_\Lambda(x'')| \leqslant M|x' - x''| \qquad (2.2)$$

for arbitrary $x', x'' \in \mathbb{R}^1$. Hence, in the thermodynamic limit we obtain that the limit function $F(x)$ [see (1.4)] obeys the Lipschitz condition

$$|F(x') - F(x'')| \leqslant M|x' - x''| \tag{2.3}$$

Using (2.3), one easily verifies that for all fixed $a \in \mathbb{R}^1$ the function $t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]$ reaches the absolute maximum with respect to $b \in \mathbb{R}^1$ on the bounded interval $|b| \leqslant 2M$. Denote by $\bar{b}(a)$ the point at which the maximum of $t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]$ is attained, and by $\bar{b}_\Lambda(a)$ the corresponding point for the function $f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]$. On the other hand, the uniform equicontinuity of the family $\{F_\Lambda(x)\}$ and the pointwise convergence (1.4) imply the uniform convergence of $\{F_\Lambda(x)\}$ to $F(x)$ on every bounded set from $\mathbb{R}^1$ (see, e.g., Ref. 14). Hence, for all $a, b \in \mathbb{R}^1$ and arbitrary fixed $D > 0$ such that $|h + \Phi_1'(a) - Lb| \leqslant D$ we find

$$|t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)] - f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]| \leqslant \delta_\Lambda(D)$$

where $t\text{-lim}\ \delta_\Lambda(D) = 0$. Thus, for every fixed $a \in \mathbb{R}^1$ one has

$$\begin{aligned} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))] &\geqslant f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))] \\ &\geqslant t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))] - \delta_\Lambda(D_a) \end{aligned} \tag{2.4}$$

where $D_a = |h| + |\Phi_1'(a)| + 2LM$. Similarly

$$\begin{aligned} t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))] &\geqslant \{t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\}|_{b=\bar{b}_\Lambda(a)} \\ &\geqslant f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))] - \delta_\Lambda(D_a) \end{aligned}$$

and, taking into account (2.4), we obtain

$$\left| \max_{b\in\mathbb{R}^1}\{t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\} - f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))] \right| \leqslant \delta_\Lambda(D_a) \tag{2.5}$$

Thus

$$\begin{aligned} &t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))] \\ &\quad\equiv \max_{b\in\mathbb{R}^1}\{t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\} \\ &\quad= t\text{-lim}\left\{\max_{b\in\mathbb{R}^1} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\right\} \equiv t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))] \end{aligned} \tag{2.6}$$

Consider now the sequence $\{f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))]\}$, $a \in \mathbb{R}^1$. In Ref. 1 we have shown that the functions $\{\bar{b}_\Lambda(a)\}$ are continuously differentiable with respect to $a \in \mathbb{R}^1$ and

$$\bar{b}_\Lambda(a) \equiv \langle A_\Lambda \rangle_{\mathscr{H}_{0,\Lambda}(a,\bar{b}_\Lambda(a))} \tag{2.7}$$

By differentiating the above identity with respect to the variable $a \in \mathbb{R}^1$ and making use of (1.11) and condition (iii) (Section 1), we conclude that

$$0 \leqslant \frac{d\bar{b}_\Lambda(a)}{da} \leqslant \frac{\Phi_1''(a)}{L} \leqslant 1 + \frac{K}{L} \tag{2.8}$$

Hence, from inequality (2.2), condition (iii), and the existence of the limit (1.4), it follows that the limit function $t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))]$ is *continuous* in $a \in \mathbb{R}^1$. Further, from the estimate

$$\begin{aligned} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))] &- f_\Lambda[\mathscr{H}_{0,\Lambda}(0, \bar{b}_\Lambda(0))] \\ &\geqslant -M(4K + 3L)|a| - \tfrac{1}{2}(L - 3K)a^2 \end{aligned} \tag{2.9}$$

in which we have taken into account the fact that $|\bar{b}_\Lambda(a)| \leqslant M$ [see (2.7) and condition (i)] as well as (2.8) and the inequality

$$a\Phi_1{}'(a) - \Phi_1(a) \geqslant \tfrac{1}{2}(L - 3K)a^2 \tag{2.10}$$

it follows that the function $t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))]$ attains its absolute minimum in the bounded interval $|a| \leqslant R = 2M(4K + 3L)/(L - 3K)$. Let $\bar{a}$, $|\bar{a}| \leqslant R$, denote the point that provides the absolute minimum value of the function $t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))]$ on $\mathbb{R}^1$. Then from (2.6) we obtain the existence of

$$\min_{a\in\mathbb{R}^1} \max_{b\in\mathbb{R}^1} \{t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\} = t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}, \bar{b}_\Lambda(\bar{a}))] \tag{2.11}$$

(*b*) Let us return now to the estimate (2.5). For all $a \in [-R, R]$ we have

$$|t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))] - f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))]| \leqslant \delta_\Lambda(\tilde{D}) \tag{2.12}$$

where $\tilde{D} = \max_{|a|\leqslant R} D_a$ is finite. The estimate (2.9) also implies the existence of the point $a = \bar{a}_\Lambda$ ($|\bar{a}_\Lambda| \leqslant R$) that provides the absolute minimum value of the function $f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}_\Lambda(a))]$ on $\mathbb{R}^1$. Therefore from (2.12) and the definition of the points $a = \bar{a}$, $a = \bar{a}_\Lambda$ we obtain

$$\begin{aligned} t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}, \bar{b}(\bar{a}))] &- f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}_\Lambda, \bar{b}_\Lambda(\bar{a}_\Lambda))] \\ &\leqslant \{t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))]\}|_{a=a_\Lambda} - f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}_\Lambda, \bar{b}_\Lambda(\bar{a}_\Lambda))] \\ &\leqslant \delta_\Lambda(\tilde{D}) \end{aligned} \tag{2.13}$$

Similarly,

$$\begin{aligned} f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}_\Lambda, \bar{b}_\Lambda(\bar{a}_\Lambda))] &- t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}, \bar{b}(\bar{a}))] \\ &\leqslant f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}, \bar{b}_\Lambda(\bar{a}))] - t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}, \bar{b}(a))] \\ &\leqslant \delta_\Lambda(\tilde{D}) \end{aligned} \tag{2.14}$$

From (2.13) and (2.14) we find

$$|f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}_N, \bar{b}_\Lambda(\bar{a}_\Lambda))] - t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(\bar{a}, \bar{b}(\bar{a}))]| \leqslant \delta_\Lambda(\tilde{D}) \tag{2.15}$$

Hence, in the thermodynamic limit we get

$$\min_{a\in\mathbb{R}^1} \max_{b\in\mathbb{R}^1} \{t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\} = t\text{-lim}\left\{\min_{a\in\mathbb{R}^1} \max_{b\in\mathbb{R}^1} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\right\} \tag{2.16}$$

which completes the proof of the lemma.

**Corollary 2.1**. If the function $F(x)$ is known, Lemma 2.1 gives the thermodynamic limit of the free energy density of the model (1.1) in terms of the two-parameter variational problem [see (1.8), (1.9) and (2.16)]

$$t\text{-}\lim f_\Lambda[H_\Lambda] = \min_{a\in\mathbb{R}^1} \max_{b\in\mathbb{R}^1}\{t\text{-}\lim f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\} \tag{2.17}$$

This result generalizes the *minimax principle* due to Bogolubov[2,3] for models with the nonpolynomial interaction (1.1).

## 3. PROOF OF THEOREM 1.1

(a) Proposition 1.1 [see (1.8)], Remark 1.2, and Lemma 2.1 imply the existence of $t$-$\lim f_\Lambda[H_\Lambda]$. The fact that $\min_{a\in\mathbb{R}^1} \max_{b\in\mathbb{R}^1} f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]$ is independent of the choice of the auxiliary parameter $L > 3K$ in Eq. (1.11) follows from (1.9).

(b) Note that the functions $\{F_\Lambda(x)\}$ and consequently the function $F(x)$ [see (1.2)–(1.4)] are *convex* on $\mathbb{R}^1$. Therefore the left derivative $F'(x - 0)$ and the right derivative $F'(x + 0)$ exist for all $x \in \mathbb{R}^1$. Hence, the condition for maximum with respect to $b \in \mathbb{R}^1$ in (2.17) is equivalent [taking into account Eq. (2.1)] to the inequalities

$$-F'(h + \Phi_1'(a) - Lb - 0) \leqslant b \leqslant -F'(h + \Phi_1'(a) - Lb + 0) \tag{3.1}$$

From the monotone nonincreasing nature of the left- and right-hand sides of (3.1) with the increase of $b \in \mathbb{R}^1$ it follows that for *each* $a \in \mathbb{R}^1$ the solution $b = \bar{b}(a)$ of inequalities (3.1) is *unique*. For $b = \bar{b}(a)$ we have

$$-F'(\bar{z}(a) - 0) \leqslant \bar{b}(a) \leqslant -F'(\bar{z}(a) + 0) \tag{3.2}$$

where

$$\bar{z}(a) \equiv h + \Phi_1'(a) - L\bar{b}(a) \tag{3.3}$$

It should be emphasized that the uniqueness of $\bar{b}(a)$ [or $\bar{b}_\Lambda(a)$, which is the solution of inequalities (3.1) with $F_\Lambda(h + \Phi_1'(a) - Lb \pm 0)$] is an immediate consequence of the *strict convexity* of the function $t$-$\lim f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]$ (or $f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]$) with respect to $b \in \mathbb{R}^1$. Furthermore, from the uniform in $b \in \mathbb{K}$ (for any compact set $\mathbb{K} \subset \mathbb{R}^1$) convergence of the sequence $\{f_\Lambda[\mathscr{H}_{0,\Lambda}(a, b)]\}$ [see Proof (a) of Lemma 2.1] and from the uniqueness of the points $\bar{b}_\Lambda(a)$ and $\bar{b}(a)$ it follows that for every $a \in \mathbb{R}^1$ one has

$$t\text{-}\lim \bar{b}_\Lambda(a) = \bar{b}(a) \tag{3.4}$$

We need now some properties of the function $\bar{b}(a)$. Integrating inequalities (2.8) over the interval $[a_1, a_2]$ and proceeding to the thermodynamic limit, we find that

$$0 \leqslant \bar{b}(a_2) - \bar{b}(a_1) \leqslant (1/L)[\Phi_1'(a_2) - \Phi_1'(a_1)] \tag{3.5}$$

i.e., $\bar{b}(a)$ is a Lipschitz-continuous [see (1.11) and condition (iii), Section 1], monotone-nondecreasing function of $a \in \mathbb{R}^1$.

Consider now the conditions for the determination of the points $\{a_n\}$ which correspond to the local minima of the function $t$- $\lim f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))]$. By definition of the point $a_m \in \{a_n\}$, there exists a neighborhood $\Sigma(a_m)$ of $a_m$ such that for all $a \in \Sigma(a_m)$

$$t\text{-}\lim f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))] - t\text{-}\lim f_\Lambda[\mathscr{H}_{0,\Lambda}(a_m, \bar{b}(a_m))] \geqslant 0$$

Hence, by using (1.12) and the concavity of $\Phi_1(a)$, it is easy to obtain the inequality

$$F(\bar{z}(a)) - F(\bar{z}(a_m)) - L\bar{b}(a_m)[\bar{b}(a) - \bar{b}(a_m)] + a[\Phi_1'(a) - \Phi_1'(a_m)] \geqslant 0 \tag{3.6}$$

Next, from the convexity of the function $F(x)$ on $\mathbb{R}^1$ it follows that for any $x_1 \leqslant x_2$

$$(x_2 - x_1)F'(x_2 - 0) \leqslant F(x_2) - F(x_1) \leqslant (x_2 - x_1)F'(x_1 + 0) \tag{3.7}$$

If $a_m \leqslant a$, $a \in \Sigma(a_m)$, then from (3.3) and (3.5) we have $\bar{z}(a_m) \leqslant \bar{z}(a)$ and from (3.6), in view of (3.2) and (3.7), we obtain

$$\begin{aligned} 0 &\leqslant [\Phi_1'(a) - \Phi_1'(a_m)][a + F'(\bar{z}(a_m) + 0)] \\ &\quad - L[\bar{b}(a) - \bar{b}(a_m)][\bar{b}(a_m) + F'(\bar{z}(a_m) + 0)] \\ &\leqslant \Phi_1''(\xi_a)(a - a_m)(a - \bar{b}(a_m)) \end{aligned} \tag{3.8}$$

where $\xi_a \in (a_m, a)$ and $\Phi_1''(\xi_a) > 0$ [see (1.11)]. Hence, for all $a \in \{a \in \Sigma(a_m): a_m \leqslant a\}$ one has

$$a \geqslant \bar{b}(a_m) \tag{3.9}$$

By similar arguments, for all $a \in \{a \in \Sigma(a_m): a \leqslant a_m\}$ [now $\bar{z}(a) \leqslant \bar{z}(a_m)$], we find

$$a \geqslant \bar{b}(a_m) \tag{3.10}$$

Combining inequalities (3.9) and (3.10), we conclude that $a_m = \bar{b}(a_m)$. We have thus proven the following important fact: Every point $a = a_m$ that corresponds to a local minimum of the function $t$-$\lim f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))]$ satisfies the equation

$$a = \bar{b}(a) \tag{3.11}$$

We observe now that on the set $S$ of all the solutions of Eq. (3.11)

$$S = \{a \in \mathbb{R}^1 : a = \bar{b}(a)\} \tag{3.12}$$

the Hamiltonians $\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))$ [(1.10)] and $H_{0,\Lambda}(a)$ [(1.5)] coincide; therefore

$$\min_{a\in\mathbb{R}^1}\{t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))]\} = \min_{a\in S}\{t\text{-lim}\, f_\Lambda[\mathscr{H}_{0,\Lambda}(a, \bar{b}(a))]\}$$
$$= \min_{a\in S}\{t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]\} \tag{3.13}$$

The definition of the set $S$ [(3.12)] can be reformulated in terms of the linearized system $\Gamma_\Lambda(x)$ [see (1.2)–(1.4)]. To this end we notice that if $\hat{a}\in S$, then from (3.2) it follows that

$$-F'(h + \varphi'(\hat{a}) - 0) \leqslant \hat{a} \leqslant -F'(h + \varphi'(\hat{a}) + 0) \tag{3.14}$$

and, conversely, if (3.14) holds, then $b = \hat{a}$ satisfies inequalities (3.1) for $a = \hat{a}$. Hence, by the uniqueness of the point $\bar{b}(a)$, we get $\bar{b}(\hat{a}) = \hat{a}$. Therefore

$$S = \{a \in \mathbb{R}^1\colon -F'(h + \varphi'(a) - 0) \leqslant a \leqslant -F'(h + \varphi'(a) + 0)\}$$

which [see (2.17) and (3.13)] completes the proof of Theorem 1.1.

**Corollary 3.1.** Let the function $\varphi(\cdot)$ in the initial Hamiltonian (1.1) correspond to an *attractive* type of interaction, i.e., let for all $a \in \Pi$

$$\varphi''(a) > 0 \tag{3.15}$$

Then

$$\min_{a\in S}\{t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]\} = \min_{a\in\mathbb{R}^1}\{t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]\}$$
$$= t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(\tilde{a})] \tag{3.16}$$

where $a = \tilde{a}$ satisfies the self-consistency equation (1.7), taken in the thermodynamic limit:

$$a = t\text{-lim}\langle A_\Lambda\rangle_{H_{0,\Lambda}(a)} \tag{3.17}$$

Actually, from the Bogolubov inequality, the spectral representation (iii) (Section 1), and condition (3.15) it follows that

$$f_\Lambda[H_{0,\Lambda}(a)] - f_\Lambda[H_\Lambda] \geqslant \tfrac{1}{2}\left\langle \int_{-M}^{M+0} dE_\lambda(A_\Lambda)\, \varphi''(\xi_\Lambda)(\lambda - a)^2 \right\rangle_{H_{0,\Lambda}(a)} \geqslant 0 \tag{3.18}$$

where $\xi_\Lambda \in (-M, M)$. Hence, taking into account that $S \subset \mathbb{R}^1$, we have

$$t\text{-lim}\, f_\Lambda[H_\Lambda] \leqslant \min_{a\in\mathbb{R}^1}\{t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]\}$$
$$\leqslant \min_{a\in S}\{t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]\} \tag{3.19}$$

Since the function $t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]$ is continuous [see (1.3)–(1.5)] and the set $S$ is bounded [$S \subset \Pi$, because $|F'(x \pm 0)| \leqslant M$; see (i) and (v), Section 1] and closed [see (3.5) and (3.12)], it reaches the minimum on some subset of the set $S$. From (3.19) and Theorem 1.1 it follows that equality (3.16) must hold for any point $\tilde{a}$ belonging to this subset. Next, taking into account (3.15) and the existence of the left and right derivatives of the function $t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]$ [see (1.3)–(1.5)], the minimum condition for $\tilde{a} \in \mathbb{R}^1$ takes the form

$$\begin{aligned} F'(h + \varphi'(\tilde{a}) - 0) + \tilde{a} &\leqslant 0 \\ F'(h + \varphi'(\tilde{a}) + 0) + \tilde{a} &\geqslant 0 \end{aligned} \tag{3.20}$$

On the other hand, by definition, $\tilde{a} \in S$. Therefore, (3.14) and (3.20) imply the differentiability of the function $t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]$ at the point $a = \tilde{a}$

$$\tilde{a} = -F'(h + \varphi'(\tilde{a})) \tag{3.21}$$

Equality (3.17) is then a consequence of the Griffiths lemma[15] about the convergence of the derivatives of the convergent sequence $\{F_\Lambda(x)\}$ of convex functions at the points of differentiability of $\{F_\Lambda(x)\}$ and the limit function $F(x)$:

$$-F'(h + \varphi'(\tilde{a})) = t\text{-lim}\{-F_\Lambda'(h + \varphi'(\tilde{a}))\} = t\text{-lim}\langle A_\Lambda \rangle_{H_{0,\Lambda}(\tilde{a})} \tag{3.22}$$

**Remark 3.1.** As was shown in Ref. 1, in the case of attractive interaction, Theorem 1.1 holds without the clustering condition (1.6). This specific property of attraction has been exploited in Ref. 16 for the particular case of $\varphi(a) = \frac{1}{2}Ja^2$, $J > 0$. The result of this paper can be generalized now to the case of an arbitrary twice differentiable function $\varphi(a)$, such that $\varphi''(a) > 0$, for $a \in \Pi$.

**Remark 3.2.** If the interaction in Hamiltonian 1.1 is not purely attractive, then the clustering property is *essential.* In the case of $\varphi(a) = \frac{1}{2}Ja^2$, $J < 0$, this question has been discussed in Ref. 17 (see also Ref. 3). Den Ouden *et al.*[7] have made an attempt to replace the clustering condition by a "short-range interaction" condition for the operators $T_\Lambda$ and $A_\Lambda$ simultaneously. In Ref. 18 it has been assumed that the bounded, self-adjoint operators $T_\Lambda$ and $A_\Lambda$ are one-particle operators, then the clustering property follows trivially.

**Corollary 3.2.** Let the function $\varphi(\cdot)$ in (1.1) correspond to a *repulsive* type of interaction, i.e., for all $a \in \Pi$ one has

$$\varphi''(a) < 0 \tag{3.23}$$

Then

$$\begin{aligned} \min_{a \in S}\{t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]\} &= \max_{a \in \mathbb{R}^1}\{t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(a)]\} \\ &= t\text{-lim}\, f_\Lambda[H_{0,\Lambda}(\bar{a})] \end{aligned} \tag{3.24}$$

where

$$\bar{a} = t\text{-lim}\, \bar{a}_\Lambda \tag{3.25}$$

and $\bar{a}_\Lambda$ is the *unique* solution of the self-consistency equation for the finite system [compare (1.7) for $S_\Lambda$]:

$$a = \langle A_\Lambda \rangle_{H_{0,\Lambda}(a)} \tag{3.26}$$

Actually, by virtue of the convexity of function $F(x)$ and condition (3.23), the set $S$ contains *only one* point $a = \bar{a}$. Hence, using Theorem 1.1, we obtain

$$\min_{a\in S}\{t\text{-}\lim f_\Lambda[H_{0,\Lambda}(a)]\} = t\text{-}\lim f_\Lambda[H_{0,\Lambda}(\bar{a})] = t\text{-}\lim f_\Lambda[H_\Lambda] \tag{3.27}$$

Next, taking into account the spectral representation (iii) (see Section 1), the Bogolubov inequality, and (3.23), we get

$$f_\Lambda[H_\Lambda] - f_\Lambda[H_{0,\Lambda}(a)] \geqslant -\tfrac{1}{2}\left\langle \int_{-M}^{M+0} dE_\Lambda(A_\Lambda)\ \varphi''(\xi_\Lambda)(\lambda - a)^2 \right\rangle_{H_\Lambda} \geqslant 0$$

where $\xi_\Lambda \in (-M, M)$. Therefore

$$\begin{aligned} t\text{-}\lim f_\Lambda[H_{0,\Lambda}(\bar{a})] &\leqslant \max_{a\in\mathbb{R}^1}\{t\text{-}\lim f_\Lambda[H_{0,\Lambda}(a)]\} \\ &\leqslant t\text{-}\lim f_\Lambda[H_\Lambda] \end{aligned} \tag{3.28}$$

Thus Eq. (3.24) is a direct consequence of (3.27) and (3.28). Equality (3.25) follows from the uniform on any bounded interval of $\mathbb{R}^1$ convergence of the sequence $\{f_\Lambda[H_{0,\Lambda}(a)]\}$ to the limit function $t$-lim $f_\Lambda[H_{0,\Lambda}(a)]$ [see Proof (a) of Lemma 2.1] and from the uniqueness [due to (3.23)] of the points $\bar{a}_\Lambda$ and $\bar{a}$.

## APPENDIX

**1.** Let the region $\Lambda \subset \mathbb{R}^\nu$ (or $\mathbb{Z}^\nu$) be of finite volume with respect to the usual Lebesgue measure on $\mathbb{R}^\nu$; $\mu(\Lambda) = |\Lambda| < \infty$ (or with respect to the corresponding discrete measure on $\mathbb{Z}^\nu$). Consider the local $C^*$-algebra of the observables $\mathfrak{A}_\Lambda$, contained in the domain $\Lambda$, that is the algebra of all bounded operators acting on the Hilbert space of states $\mathfrak{H}_\Lambda$.[6] If $x \in \Lambda$, then the operator-valued function $A: x \to A(x) \in \mathfrak{A}_\Lambda$ is called a *local observable* (*local operator*). Along with this it is convenient to define "quasilocal quantities" (Haag[4]). Let the continuous function $f_Q(x, y)$ be such that there exists $Q > 0$ and $f_Q(x, y) = 0$ for $|x - y| > Q$; then

$$A_Q(y) = \int_\Lambda dx\, f_Q(x, y) A(x)$$

is called a *quasilocal operator*.

The *space average* of the local (quasilocal) operator $A(x)$ over the region $A \subset \mathbb{R}^\nu$ (or $\mathbb{Z}^\nu$) is defined for arbitrary $x \in \Lambda$ as

$$A_\Lambda = (1/|\Lambda|) \int_{G_\Lambda(x)} dy\ \tau_y A(x) \tag{A.1}$$

where the translation $\tau_y$ acts on the operators $A(x)$ as follows: $\tau_y A(x) =$

$A(x + y)$. The operator $A_\Lambda$ is called an *intensive*[1] (or *normalized*[7]) operator. Here $G_\Lambda(x)$ is such that for all $\tau_y \in G_\Lambda(x)$ we have $x + y \in \Lambda$. For $\mathbb{Z}^\nu$ the corresponding discrete measure $dy$ induces summation over the sublattice $\Lambda \subset \mathbb{Z}^\nu$. A similar construction for $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots$ and $|\Lambda_k| \to \infty$ is called an "averaging operation,"[19] $M$-filter,[20] or $M$-net[6] (see also Ref. 21).

**2.** With the notion of the space-average (or $M$-filter, $M$-net) of quasilocal operators one can formulate such a property of the infinite system states $\rho(\cdot)$ as *weak clustering*[19–21]:

$$\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \int_{G_\Lambda(x)} dy\, \rho(\tau_y A(x) B) = \rho(A(x))\rho(B) \tag{A.2}$$

for arbitrary $A(x)$, $B \in \bigcup_{\Lambda \subset \mathbb{R}^\nu} \mathfrak{A}_\Lambda$. This property is necessary for the $G$-invariant state $\rho(\cdot)$ to correspond to a *pure phase* (see Refs. 6 and 19–21).

In the present work we have used a *clustering property* [see (vi), Section 1], which is obviously *weaker* than (A.2), since (vi) involves *only one* intensive operator, the one appearing in interaction Hamiltonian (1.1). This means that for such a model the infinite system states generated by the approximating Hamiltonian (1.5) may not correspond to pure phases. Thus, the condition (vi) is just a restriction on the fluctuations of the intensive operator $A_\Lambda$.

A trivial example, when the clustering property (vi) occurs, corresponds to the case of one-particle operators $T_\Lambda$ and $A_\Lambda$ (see Refs. 2, 3, 18, and 22). It can easily be verified that the infinite system states generated by the approximating Hamiltonian (1.5) for all $a \in \mathbb{R}^1$ are $G$-invariant and weakly clustering.

Now, let $\Lambda \subset \mathbb{Z}^2$, $|\Lambda| < \infty$, and let the operator

$$T_\Lambda = -\frac{J}{2} \sum_{|i-j|=1} \sigma_i \sigma_j, \qquad J > 0$$

describe the square Ising model ($\sigma = \pm 1$) with nearest neighbor interaction. Let the space average $A_\Lambda$ be

$$A_\Lambda = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} A(i)$$

where $A(0)$ denotes the quasilocal operator $(\sigma_0/2) \sum_{|0-j|=1} \sigma_j$ and

$$A(i) = \tau_i A(0) = \tfrac{1}{2}\sigma_i \sum_{|i-j|=1} \sigma_j$$

Then the infinite system states generated by the approximating Hamiltonian (1.5) are known to be not weakly clustering for some domain of the variables

$\beta > 0$, $h \in \mathbb{R}^1$, and $a \in \mathbb{R}^1$. Nevertheless, the clustering property (vi) holds because the fluctuations in (1.6) are proportional to $|\Lambda|^{-1}c_\Lambda(\beta, a, h)$, where $c_\Lambda(\beta, a, h)$ is the specific heat capacity, which according to Ref. 23 is bounded above by $O(\ln|\Lambda|)$ for $|\Lambda| \to \infty$.

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